

look at :

$$\begin{cases} \tilde{x}(t) = \tilde{x}(0) \cos \omega t + \frac{\tilde{p}(0)}{m\omega} \sin \omega t \\ \tilde{p}(t) = -m\omega \tilde{x}(0) \sin \omega t + \tilde{p}(0) \cos \omega t \end{cases}$$

$$\Leftrightarrow m \frac{d\tilde{x}}{dt} = \tilde{p}, \text{ just like C.M.}$$

But, $\langle \tilde{x} \rangle$ and $\langle \tilde{p} \rangle$ are "not" oscillating. !!!

although $\tilde{x}(t)$ and $\tilde{p}(t)$ look like oscillating.

|| NOTE: $\langle \tilde{x} \rangle = 0$, $\langle \tilde{p} \rangle = 0$, for all $|n\rangle$.

Q. Can we find a "Quantum" state that behaves just like classical $\langle x \rangle$ and $\langle p \rangle$?

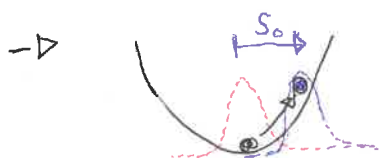
* Coherent States

← This is the one.

Why do we need this?

- We live in a "classical" world,
But we want to control a "Quantum" world.
∴ We need a "bridge"!

the easiest way to make the coherent state



$$|S_0\rangle = J(S_0) |0\rangle$$

move the ground state to S_0 .

$$\text{Wave function } \psi_{S_0}(x) = \psi_0(x - S_0) \quad \left\| \begin{aligned} \langle x | J(S_0) \\ = \langle x - S_0 | \end{aligned} \right.$$

observables :

$$\langle S_0 | \tilde{x} | S_0 \rangle = \langle 0 | J^\dagger(S_0) \tilde{x} J(S_0) | 0 \rangle = \underline{S_0}$$

$$\langle S_0 | \tilde{p} | S_0 \rangle = 0$$

$$\langle S_0 | H | S_0 \rangle = \langle 0 | \frac{\tilde{p}^2}{2m} | 0 \rangle + \frac{1}{2} m \omega^2 \langle 0 | (\tilde{x} + S_0)^2 | 0 \rangle$$

(Cont.)

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$$\langle s_0 | H | s_0 \rangle = \langle 0 | H | 0 \rangle + \frac{1}{2} m \omega^2 s_0^2$$

$$= \underbrace{\frac{1}{2} \hbar \omega}_{\text{"quantum."}} + \underbrace{\frac{1}{2} m \omega^2 s_0^2}_{\text{due to the displacement.}}$$

NOTE: $|s_0\rangle$ is not an energy eigenstate!

• Time evolution: $|s_0\rangle \longrightarrow |s_0, t\rangle$

• $\langle s_0, t | \tilde{x} | s_0, t \rangle = \langle s_0 | \tilde{x}(t) | s_0 \rangle$ ↖ Heisenberg picture.

$$= \langle s_0 | \tilde{x}(0) \cos \omega t + \frac{\tilde{p}(0)}{m\omega} \sin \omega t | s_0 \rangle$$

$$= s_0 \cos \omega t.$$

$$\bullet \langle s_0, t | \tilde{p} | s_0, t \rangle = \langle s_0 | \tilde{p}(t) | s_0 \rangle$$

$$= -m\omega s_0 \sin \omega t.$$

∴ $\langle \tilde{x} \rangle_{s_0}$ and $\langle \tilde{p} \rangle_{s_0}$ are oscillating
as their classical counterparts do.

$$\begin{aligned} \bullet \langle (\Delta \tilde{x})^2 \rangle_{s_0} &= \langle s_0 | \tilde{x}(t)^2 | s_0 \rangle - \langle s_0 | \tilde{x}(t) | s_0 \rangle^2 \\ &= \langle s_0 | \tilde{x}(0)^2 \cos^2 \omega t + \frac{\tilde{p}(0)^2}{(m\omega)^2} \sin^2 \omega t + \underbrace{\mathcal{O}[\tilde{x}, \tilde{p}]}_{\langle [\tilde{x}, \tilde{p}] \rangle_{s_0} = 0} | s_0 \rangle \\ &\quad - s_0^2 \cos^2 \omega t \end{aligned}$$

$$= \left(\cancel{s_0^2} + \frac{\hbar}{2m\omega} \right) \cos^2 \omega t + \frac{m\hbar\omega}{2} \cdot \frac{1}{(m\omega)^2} \sin^2 \omega t - \cancel{s_0^2 \cos^2 \omega t}$$

$$= \frac{\hbar}{2m\omega}$$

(t -indep.)

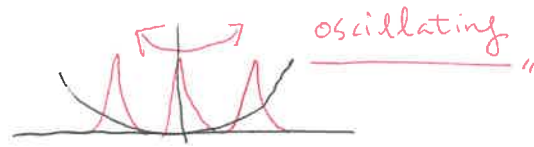
∴ It's the same
as in 107 !!!

$$\langle (\Delta \tilde{p})^2 \rangle_{s_0} = \frac{m\hbar\omega}{2}$$

$$\Rightarrow \langle (\tilde{x})^2 \rangle_{s_0} \langle (\tilde{p})^2 \rangle_{s_0} = \frac{\hbar^2}{4} \quad \text{minimum uncertainty!}$$

(Gaussian wave packet)

So, It looks like



$|s_0\rangle$ in the energy eigenkets as base kets.

$$|s_0\rangle = \exp\left(-\frac{\tilde{p} \tilde{s}_0}{\hbar}\right) |0\rangle = \exp\left[\frac{s_0}{\sqrt{2}x_0}(\tilde{a}^\dagger - \tilde{a})\right] |0\rangle$$

$$\tilde{p} = \tilde{p} \sqrt{\frac{m\hbar\omega}{2}} (-\tilde{a} + \tilde{a}^\dagger) \quad \parallel \quad x_0 = \sqrt{\frac{\hbar}{m\omega}}$$

using the case in the Baker-Campbell-Hausdorff theorem.

$$[e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}]$$

provided that $[A, [A, B]] = [B, [A, B]] = 0$

$$|s_0\rangle = e^{\frac{s_0}{\sqrt{2}x_0} \tilde{a}^\dagger} e^{-\frac{s_0}{\sqrt{2}x_0} \tilde{a}} e^{-\frac{1}{4} \frac{s_0^2}{x_0^2}} |0\rangle$$

$1 + O(\tilde{a})$ C-numbers
 $\rightarrow 0$ with $|0\rangle$

$$= e^{-\frac{1}{4} \frac{s_0^2}{x_0^2}} \exp\left[\frac{s_0}{\sqrt{2}x_0} \tilde{a}^\dagger\right] |0\rangle$$

$$\Rightarrow |s_0\rangle = e^{-\frac{1}{4} \frac{s_0^2}{x_0^2}} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{s_0}{\sqrt{2}x_0}\right)^n (\tilde{a}^\dagger)^n |0\rangle$$

$$= e^{-\frac{1}{4} \frac{s_0^2}{x_0^2}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}} \left(\frac{s_0}{\sqrt{2}x_0}\right)^n |n\rangle$$

$$\equiv \sum_{n=0}^{\infty} c_n |n\rangle \quad \parallel \quad c_n = e^{-\frac{1}{4} \frac{s_0^2}{x_0^2}} \frac{1}{\sqrt{n!}} \left(\frac{s_0}{\sqrt{2}x_0}\right)^n$$

$$\begin{aligned} \text{prob. for } |s_0\rangle &= |C_n|^2 \\ \text{to be } |n\rangle &= \exp\left[-\frac{1}{2} \frac{s_0^2}{x_0^2}\right] \cdot \frac{1}{n!} \left(\frac{s_0}{\sqrt{2}x_0}\right)^{2n} \end{aligned}$$

$$\equiv e^{-\bar{n}} \frac{\bar{n}^n}{n!} \quad \parallel \quad \bar{n} = \frac{s_0^2}{2x_0^2}$$

← This is the Poisson distribution, with mean \bar{n} .

$$\begin{aligned} \bar{n} = \text{expectation value of } n &= \sum_{n=0}^{\infty} n |C_n|^2 = \sum_{n=0}^{\infty} n e^{-\bar{n}} \frac{\bar{n}^n}{n!} \\ &= e^{-\bar{n}} \bar{n} \frac{d}{d\bar{n}} \left(\sum_{n=0}^{\infty} \frac{\bar{n}^n}{n!} \right) \\ &= e^{-\bar{n}} \bar{n} \frac{d}{d\bar{n}} (e^{\bar{n}}) = \bar{n} \end{aligned}$$

$$\langle s_0 | \tilde{N} | s_0 \rangle = \sum_{n,m} C_n^* C_m n \delta_{n,m} = \bar{n}.$$

$$\Rightarrow \langle E \rangle = \hbar \omega \left(\bar{n} + \frac{1}{2} \right) \approx \hbar \omega \bar{n} \quad \parallel \quad s_0^2 \gg x_0^2$$

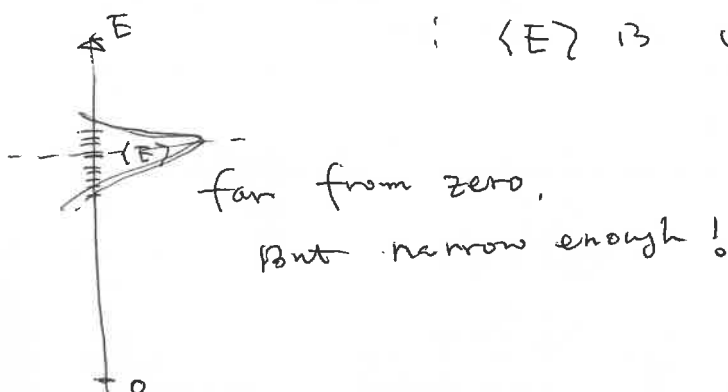
• Energy uncertainty

$$\Delta E = \hbar \omega \left[\overline{n^2} - \bar{n}^2 \right]^{\frac{1}{2}} = \hbar \omega \frac{s_0}{2x_0}$$

$$\Rightarrow \frac{\Delta E}{\hbar \omega} = \frac{s_0}{2x_0} \gg 1. \quad (\text{classical limit})$$

$$\text{but, } \frac{\langle E \rangle}{\Delta E} = \frac{s_0}{2x_0} \gg 1$$

∴ $\langle E \rangle$ is well defined!



• Generalization

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$$|s_0\rangle = \exp\left(-\frac{i\tilde{p}s_0}{\hbar}\right)|0\rangle = \exp\left[\frac{s_0}{\sqrt{2}x_0}(\tilde{a}^\dagger - \tilde{a})\right]|0\rangle$$

↳ generalization

$$|\alpha\rangle = D(\alpha)|0\rangle = \exp(\alpha\tilde{a}^\dagger - \alpha^*\tilde{a})|0\rangle$$

↑
unitary

↑
It generates a "displacement"
in "generalized" coordinates.

what's $|\alpha\rangle$?

$$\tilde{a}|\alpha\rangle = \alpha|\alpha\rangle \quad \text{eigenket of } \tilde{a}.$$

proof.

$$\tilde{a}|\alpha\rangle = \tilde{a} \exp(\alpha\tilde{a}^\dagger - \alpha^*\tilde{a})|0\rangle$$

$$= [\tilde{a}, e^{\alpha\tilde{a}^\dagger - \alpha^*\tilde{a}}]|0\rangle \quad \parallel \tilde{a}|0\rangle = 0$$

$$= \alpha|\alpha\rangle$$

$$\text{using } [A, e^B] = [A, B]e^B$$

$$\text{when } [[A, B], B] = 0.$$

→ expectation values

$$\langle\alpha|\tilde{x}|\alpha\rangle = \frac{x_0}{\sqrt{2}} \langle\alpha|\tilde{a} + \tilde{a}^\dagger|\alpha\rangle = \frac{x_0}{\sqrt{2}} (\alpha + \alpha^*) = \sqrt{2}x_0 \text{Re}[\alpha]$$

$$\langle\alpha|\tilde{p}|\alpha\rangle = \sqrt{2}\hbar x_0^{-1} \text{Im}[\alpha]$$

→ Time-evolution

$$|\alpha, t\rangle = e^{-\frac{i\tilde{H}t}{\hbar}} e^{\alpha\tilde{a}^\dagger - \alpha^*\tilde{a}} e^{\frac{i\tilde{H}t}{\hbar}} e^{-\frac{i\tilde{H}t}{\hbar}}|0\rangle$$

$$= e^{\alpha\tilde{a}^\dagger(-t) - \alpha^*\tilde{a}(-t)} e^{-\frac{i\hbar\omega t}{2}}|0\rangle$$

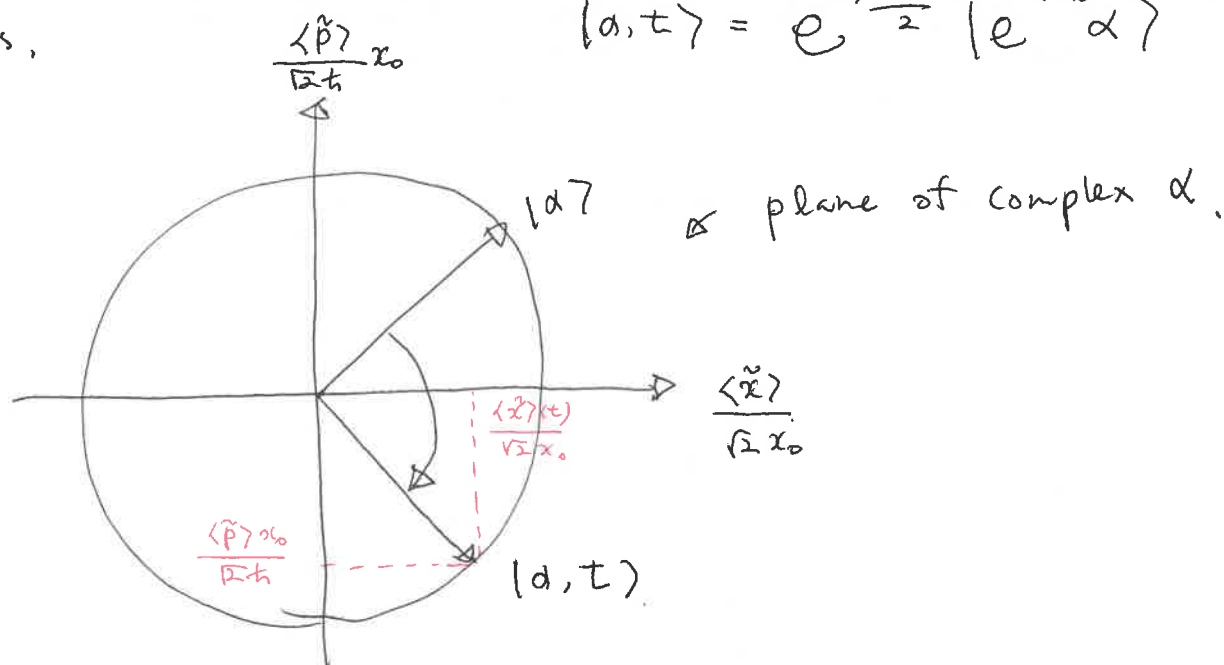
$$\text{and } \begin{aligned} \tilde{a}^\dagger(-t) &= e^{-i\omega t} \tilde{a}^\dagger \\ \tilde{a}(-t) &= e^{i\omega t} \tilde{a} \end{aligned} \quad (\alpha e^{-i\omega t}) \tilde{a}^\dagger - (\alpha e^{-i\omega t})^* \tilde{a}$$

thus,

$$|\alpha, t\rangle = e^{-\frac{i\hbar\omega t}{2}} |e^{-i\omega t}\alpha\rangle$$

Thus,

$$|a, t\rangle = e^{-\frac{i\omega t}{2}} |e^{-i\omega t} \alpha\rangle \quad 31$$



\longleftrightarrow "classical" phase diagram
in the generalized coordinates. (p. 3)

Supplement

• properties of $D(\alpha)$

$$① \quad D^\dagger(\alpha) = D^{-1}(\alpha) = D(-\alpha) \quad : \text{unitarity}$$

$$② \quad D^\dagger(\alpha) \tilde{a} D(\alpha) = \tilde{a} + \alpha$$

$$③ \quad D^\dagger(\alpha) \tilde{a}^\dagger D(\alpha) = \tilde{a}^\dagger + \alpha^*$$

$$④ \quad D(\alpha + \beta) = D(\alpha) D(\beta) \cdot \exp[-\pi \operatorname{Im}(\alpha \beta^*)]$$

• $\{|\alpha\rangle\}$ is non-orthogonal basis.

$$|\langle \alpha | \beta \rangle| = \exp[-|\alpha - \beta|^2] \neq 0 \quad \text{for } \alpha \neq \beta.$$

• still, $\{|\alpha\rangle\}$ has a completeness relation

(overcomplete!)

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle \langle \alpha| = \mathbb{1}$$